Math 10B with Professor Stankova
Worksheet, Discussion #19; Thursday, 4/4/2019
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Standard Deviation

Concepts

1. The **variance** of a random variable is defined as $E[(X-\mu)^2]$ and there is a shortcut formula that we can use to define it as $E[X^2]-\mu^2$. For continuous random variables, we replace summation with

$$\sigma^2 = E[X^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$

Example

2. Let $f(x) = e \cdot e^x$ for $x \leq -1$ and 0 otherwise. Find the standard deviation of this distribution.

Solution: First we need to find the mean of this distribution. The mean is

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} x(e \cdot e^x)dx + \int_{-1}^{\infty} 0dx = e\int_{-\infty}^{-1} xe^x dx$$

$$= e(xe^x - e^x|_{-\infty}^{-1}) = e[(-e^{-1} - e^{-1}) - 0] = -2.$$

To find the standard deviation, we first find the variance and then take the square root. There are two ways to do this, the latter is a bit easier

$$\sigma^2 = \int_{-\infty}^{\infty} (x-(-2))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-\infty}^{-1} x^2(e \cdot e^x)dx - 4$$

$$= e(x^2e^x - 2xe^x + 2e^x|_{-\infty}^{-1}) - 4 = e(e^{-1} + 2e^{-1} + 2e^{-1}) - 4 = 5 - 4 = 1.$$

So the standard deviation is $\sigma = 1$.

Problems

3. True **FALSE** The standard deviation always exists.

Solution: The standard deviation requires the mean to exist, and sometimes that doesn't exist.

4. True **FALSE** Sometimes, we take the standard deviation to be the negative square root of the variance.

Solution: The standard deviation is always nonnegative.

5. **TRUE** False The variance is always nonnegative.

Solution: The variance is $\int (x-\mu)^2 f(x)dx$ and both $(x-\mu)^2 \geq 0$ and $f(x) \geq 0$ so $(x-\mu)^2 f(x) \geq 0$ so the integral must be nonnegative too.

6. **TRUE** False If the mean doesn't exist, then the standard deviation doesn't exist.

Solution: The formula for the standard deviation requires the mean, so if the mean doesn't exist, then we can't talk about the standard deviation.

7. True **FALSE** If the mean exists, then the standard deviation exists.

Solution: It is possible for the mean to exist but the standard deviation not to exist. For example, the distribution $\frac{1}{x^3}$ on $x \geq 1$ has the mean existing but the standard deviation not.

8. Let $f(x)$ be $2/3x$ from $1 \leq x \leq 2$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_1^2 2/3x^2 dx = \frac{2}{9}x^3|_1^2 = \frac{14}{9}.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x - 14/9)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_1^2 2/3x^3 dx - (14/9)^2 = \frac{5}{2} - \frac{196}{81} = \frac{13}{162}.$$

Thus, $\sigma = \sqrt{13/162} = \sqrt{26}/18$.

9. Let $f(x)$ be $-4/x^5$ for $x \leq -1$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} -4/x^4 dx = \frac{4}{3}x^{-3}|_{-\infty}^{-1} = \frac{-4}{3}.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-(-4/3))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-\infty}^{-1} -4/x^3 dx - (-4/3)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

Thus, $\sigma = \sqrt{2/9} = \sqrt{2}/3$.

10. Let $f(x)$ be the uniform distribution on $0 \leq x \leq 10$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: Since f is the uniform distribution on $[0, 10]$, we know that $f(x) = \frac{1}{10-0} = \frac{1}{10}$ on $[0, 10]$ and 0 everywhere else. First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{10} x/10 dx = x^2/20|_0^{10} = 5.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-5)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{0}^{10} x^2/10 dx - 5^2 = \frac{100}{3} - 25 = \frac{25}{3}.$$

Thus, $\sigma = \sqrt{25/3} = 5\sqrt{3}/3$.